

CONDITION NUMBER ESTIMATES FOR MATRICES ARISING IN NURBS BASED ISOGEOMETRIC DISCRETIZATIONS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract. We derive bounds for the minimum and maximum eigenvalues and the spectral condition number of matrices for isogeometric discretizations of elliptic partial differential equations in an open, bounded, simply connected Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$. We consider refinements based on mesh size h and polynomial degree p with maximum regularity of spline basis functions. For the h -refinement, the condition number of the stiffness matrix is bounded above by a constant times h^{-2} and the condition number of the mass matrix is uniformly bounded. For the p -refinement, the condition number grows exponentially and is bounded above by $p^{2d+2}4^{pd}$ and $p^{2d}4^{pd}$ for the stiffness and mass matrices, respectively. Rigorous theoretical proofs of these estimates and supporting numerical results are provided.

Key words. Elliptic PDEs, Galerkin formulation, B-Splines, NURBS, Isogeometric method, Stiffness matrix, Mass matrix, h -refinement, p -refinement, Eigenvalues, Condition number

AMS subject classifications.

1. Introduction. Isogeometric analysis is a term introduced by Hughes et al. in 2005 [28]. Most of the research activity in isogeometric analysis has focused on using Non-Uniform Rational B-Spline (NURBS) as basis functions, e.g., [1, 5, 9, 28]. Isogeometric analysis is not restricted to NURBS basis functions. Other types of basis functions are used by researchers, e.g., T-Splines, hierarchical B-Splines, and subdivision schemes. Use of splines as finite element basis functions dates back to the 1970's, however [41, 46, 50].

In isogeometric analysis the computational geometry (e.g., a circle) is represented exactly from the information and the basis functions given by Computer Aided Design (CAD). It holds an advantage over classical finite element methods (FEM), where the basis functions are defined using piecewise polynomials and the computational geometry (i.e., a mesh) is defined on polygonal elements. It has been argued in [9] that NURBS based isogeometric method leads to qualitatively more accurate results than a standard piecewise polynomial based finite element method. Typically, the solution computed by an isogeometric method has a higher continuity than the one computed in a classical finite element method. It is a difficult and cumbersome task to achieve even C^1 inter-element continuity in the piecewise polynomial based finite element method, whereas isogeometric method offers up to C^{p-m} continuity, where p denotes the degree of the basis functions and m denotes the knot-multiplicity. Finally, isogeometric analysis provides a powerful tool to compute highly continuous numerical solution of PDEs arising in engineering sciences.

*This research was supported in part by the Austrian Sciences Fund (Project P21516-N18), National Science Foundation grant EPS-1135483, Award No. KUS-C1-016-04, made by King Abdullah University of Science & Technology (KAUST), and the KAUST Numerical Porous Media Center.

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Since the introduction of isogeometric analysis, most of its progress has been focused on the applications and discretization properties. Nevertheless, when dealing with large problems, the cost of solving the linear system of equations arising from the isogeometric discretization becomes an important issue. Clearly, the discretization matrix A gets denser by increasing the polynomial degree p . Therefore, the cost of a direct solver, particularly for large problems, becomes prohibitively expensive. The most practical way to solve them is to resort to an iterative method. Since the convergence rate of such methods is strongly affected by the condition number of the system matrix A , it is important to assess this quantity as a function of the mesh size h for the h -refinement, or as a function of the degree p for the p -refinement. Note that in the p -refinement, improved approximate solutions are sought by increasing p while the mesh of the domain, and thus the maximum quadrilateral diameter h , is held fixed, whereas in the h -refinement, improved approximations are obtained by refining the mesh, and thus reducing h , while p is held fixed. In this paper we consider both the cases: h - and p -refinements. Similar efforts are made in [22] on the spectrum of stiffness matrices and in [40] on bounding the influence of the domain parameterization and knot spacing. However, these papers primarily derive bounds with respect to the mesh size h . To the best of our knowledge there is no study that discusses the bounds on condition number estimates of isogeometric matrices with respect to p -refinement. Our main results provide upper bounds for the condition number of the stiffness matrix and the mass matrix for both the h - and p -refinements.

For h -refinement applied to second order elliptic problems on a regular mesh, the condition number of the finite element stiffness matrix scales as h^{-2} and the condition number of the mass matrix is bounded uniformly, independent of h [3]. This is true for a great variety of elements and independent of the dimension of the problem domain. Our results here are in agreement [2] and are useful in theoretical analysis that relates to h -refinement. For example, in convergence analysis of multigrid methods, these results are one of the key elements in deriving convergence factors, for finite element analysis [7, 23, 43] and for isogeometric analysis [20].

The order of the approximation error of the numerical solution depends on the choice of the finite dimensional subspace, not on the choice of its basis [8]. Therefore, when working with a finite element method or an isogeometric method for elliptic problems, we only consider function spaces rather than the choice of particular basis functions. Nevertheless, the choice of the basis functions affects the condition number of the stiffness and the mass matrices, which influences the performance of iterative solvers. There is no general theory to characterize the extremal eigenvalues or the condition number based on a set of general polynomial basis functions [4, 33, 34, 35]. Unlike the h -refinement case, the condition number heavily depends on the choice of basis functions for the p -refinement.

For different choices of basis functions the condition number may grow algebraically or exponentially. Olsen and Douglas [37] estimated the condition number bounds of finite element matrices for tensor product elements with two choices of basis functions. For Lagrange elements, it is proved that the condition number grows exponentially in p . For hierarchical basis functions based on Chebychev polynomials, the condition number grows rapidly but only algebraically in p . Similar results on the condition number bounds can be found in [18, 27, 32].

Due to the larger support of NURBS basis functions, the band of the stiffness matrix corresponding to the NURBS-based isogeometric method is less sparse than the one arising from piecewise polynomial finite element procedures. Therefore, a larger

condition number is expected. Our results for the p -refinement case show that the condition number of system matrices in an isogeometric method grows exponentially.

Throughout this paper we deal with the maximum regularity C^{p-1} of a B-spline unless otherwise specified. The generic constant C , which will be used often takes different values at different occasions, and is independent of h and p in the analysis with respect to h -refinement and p -refinement, respectively. Moreover, in our numerical studies the coarsest and finest meshes use $h = 1$ and $h = 1/128$, respectively.

The remainder of the paper is organized as follows. In Section 2, we describe the model problem and its discretization. In Section 3, we define B-Splines and NURBS and a basic notation. We recall bounds for the condition number of a B-Spline basis function. In Section 4, we derive bounds for the eigenvalues and the condition number of the stiffness and mass matrices arising in isogeometric discretizations for the h - and p -refinement cases. In Section 5, we provide numerical experiments that support the theoretical estimates. In Section 6, we draw some conclusions and discuss future work.

2. Model problem and its discretization. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be an open, bounded, and simply connected Lipschitz domain with Dirichlet boundary $\partial\Omega$. We consider the Poisson equation,

$$(2.1a) \quad \Delta u = -f \quad \text{in } \Omega,$$

$$(2.1b) \quad u = 0 \quad \text{on } \partial\Omega,$$

where $f : \Omega \rightarrow \mathbb{R}$ is given. The aim is to find $u : (\Omega \cup \partial\Omega) \rightarrow \mathbb{R}$ that satisfies (2.1). We consider Galerkin's formulation of the problem, which is commonly used in isogeometric analysis. Since we are interested in the study of the condition number, therefore we shall not go into the details of the solution properties, and restrict ourselves to the study of the condition number of the resulting system matrices.

Isogeometric analysis has the same theoretical foundation as finite element analysis, namely the variational form of a partial differential equation. We define the function space \mathcal{S} as all the functions that have square integrable derivatives and also satisfy $u|_{\partial\Omega} = 0$,

$$(2.2) \quad \mathcal{S} = \{u : u \in H^1(\Omega), u|_{\partial\Omega} = 0\},$$

where $H^1(\Omega) = \{u : D^\alpha u \in L^2(\Omega), |\alpha| \leq 1\}$ is a Sobolev space, $\alpha \in \mathbb{N}^d$ is a multi-index, $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_d^{\alpha_d}$, and $D_i^j = \frac{\partial^j}{\partial x_i^j}$.

We write the variational formulation of the model problem by multiplying it by an arbitrary function $v \in \mathcal{S}$ and integrating by parts. For a given f : find $u \in \mathcal{S}$ such that for all $v \in \mathcal{S}$,

$$\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega = \int_{\Omega} f v \, d\Omega.$$

We rewrite the formulation as: find $u \in \mathcal{S}$ such that for all $v \in \mathcal{S}$,

$$(2.3) \quad a(u, v) = L(v), \quad \forall v \in \mathcal{S},$$

where

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, d\Omega, \quad \text{and} \quad L(v) = \int_{\Omega} f v \, d\Omega.$$

Note that $a(\cdot, \cdot)$ is a bilinear form that is continuous and coercive on \mathcal{S} . $L(\cdot)$ is a linear form associated with the original equation.

Let $\mathcal{S}^h \subset \mathcal{S}$ be a finite dimensional approximation of \mathcal{S} . The Galerkin form of the problem is: Find $u^h \in \mathcal{S}^h$ such that for all $v^h \in \mathcal{S}^h$,

$$(2.4) \quad a(u^h, v^h) = L(v^h),$$

which is a well-posed problem with a unique solution [8].

By approximating u_h and v_h using spline (see Section 3) basis functions N_i , $i = 1, 2, \dots, n_h = \mathcal{O}(h^{-2})$, the variational formulation (2.4) is transformed into a set of linear algebraic equations,

$$(2.5) \quad \mathbf{A}\mathbf{u} = \mathbf{f}.$$

\mathbf{A} denotes the stiffness matrix obtained from the bilinear form $a(\cdot, \cdot)$,

$$A = (a_{i,j}) = (a(N_i, N_j)), \quad i, j = 1, 2, 3, \dots, n_h.$$

\mathbf{u} denotes the vector of unknown degrees of freedom and \mathbf{f} denotes the right hand side vector from the known data of the problem. \mathbf{A} is a real, symmetric positive definite matrix.

3. Splines and their condition number bounds. Non-uniform rational B-Splines (NURBS) are commonly used in isogeometric analysis and are built from B-Splines. In Section 3.1, we give a brief description of B-Splines and NURBS and their properties. In Section 3.2, we define the derivatives of B-Splines. In Section 3.3, we prove bounds on the condition number of B-Spline basis functions.

3.1. B-Splines and NURBS. In this section, we define B-Spline and NURBS functions. We also define surfaces and describe higher order objects based on both types of functions.

The Cox-de Boor recursion formula [13] is given by

DEFINITION 3.1. Let $\Xi_1 = \{\xi_i : i = 1, \dots, n + p + 1\}$ be a non-decreasing sequence of real numbers called the knot vector, where ξ_i is the i^{th} knot, p is the polynomial degree, and n is the number of basis function. With a knot vector in hand, the B-Spline basis functions denoted by $N_i^p(\xi)$ are (recursively) defined starting with a piecewise constant ($p = 0$):

$$(3.1a) \quad N_i^0(\xi) = \begin{cases} 1 & \text{if } \xi \in [\xi_i, \xi_{i+1}), \\ 0 & \text{otherwise,} \end{cases}$$

$$(3.1b) \quad N_i^p(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_i^{p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1}^{p-1}(\xi),$$

where $0 \leq i \leq n$, $p \geq 1$ and $\frac{0}{0}$ is considered as zero.

For a B-Spline basis function of degree p , an interior knot can be repeated at most p times, and the boundary knots can be repeated at most $p + 1$ times. A knot vector for which the two boundary knots are repeated $p + 1$ times is said to be open. In this case, the basis functions are interpolatory at the first and the last knot. Important properties of the B-Spline basis functions include nonnegativity, partition of unity, local support and C^{p-k} -continuity.

Higher dimensional B-Spline objects are defined using tensor products.

DEFINITION 3.2. A B-Spline curve $C(\xi)$ is defined by

$$(3.2) \quad C(\xi) = \sum_{i=1}^n P_i N_i^p(\xi),$$

where $\{P_i : i = 1, \dots, n\}$ are the control points and N_i^p are B-Spline basis functions defined in (3.1).

DEFINITION 3.3. A B-Spline surface $S(\xi, \eta)$ is defined by

$$(3.3) \quad S(\xi, \eta) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} N_{i,j}^{p_1, p_2}(\xi, \eta) P_{i,j},$$

where $P_{i,j}$, $i = 1, 2, \dots, n_1$, $j = 1, 2, \dots, n_2$, denote the control points, $N_{i,j}^{p_1, p_2}$ is the tensor product of B-Spline basis functions $N_i^{p_1}$ and $N_j^{p_2}$, and $\Xi_1 = \{\xi_1, \xi_2, \dots, \xi_{n_1+p_1+1}\}$ and $\Xi_2 = \{\eta_1, \eta_2, \dots, \eta_{n_2+p_2+1}\}$ are the corresponding knot vectors.

Similarly three dimensional B-Spline solids can be defined using two tensor products.

Polynomials cannot exactly describe frequently encountered shapes in engineering, particularly the conic family (e.g., a circle). While B-Splines are flexible and have many nice properties for curve design, they are also incapable of representing such curves exactly. Such limitations are overcome by NURBS functions that can exactly represent a wide array of objects.

Rational representation of conics originates from projective geometry. The “coordinates” in the additional dimension are called weights, which we shall denote by w . Furthermore, let $\{P_i^w\}$ be a set of control points for a projective B-Spline curve in \mathbb{R}^3 . For the desired NURBS curve in \mathbb{R}^2 , the weights and the control points are derived by the relations

$$(3.4) \quad w_i = (P_i^w)_3, \quad (P_i)_d = (P_i^w)/w_i, \quad d = 1, 2,$$

where w_i is called the i^{th} weight and $(P_i)_d$ is the d^{th} -dimension component of the vector P_i . The weight function $w(\xi)$ is defined as

$$(3.5) \quad w(\xi) = \sum_{i=1}^n N_i^p(\xi) w_i.$$

We now formally define NURBS objects.

DEFINITION 3.4. The NURBS basis functions and curve are defined by

$$(3.6) \quad R_i^p(\xi) = \frac{N_i^p(\xi) w_i}{w(\xi)} \quad \text{and} \quad C(\xi) = \sum_{i=1}^n R_i^p(\xi) P_i.$$

DEFINITION 3.5. The NURBS surfaces are defined by

$$(3.7) \quad S(\xi, \eta) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} R_{i,j}^{p_1, p_2}(\xi, \eta) P_{i,j},$$

where $R_{i,j}^{p_1, p_2}$ is the tensor product of NURBS basis functions $R_i^{p_1}$ and $R_j^{p_2}$. NURBS functions also satisfy the properties of B-Spline functions [39, 42, 47].

3.2. Derivatives of B-Splines. Derivatives of B-Splines [19] and their conditioning are very important for the estimation of the condition number of the stiffness matrix. The recursive definition of B-Spline functions allow us to seek the relationship between the derivative of a given B-Spline basis function and lower degree basis function.

DEFINITION 3.6. *The derivative of the i^{th} B-Spline basis function defined in (3.1) is given by*

$$(3.8) \quad \frac{d}{d\xi} N_i^p(\xi) = \frac{p}{\xi_{i+p} - \xi_i} N_i^{p-1}(\xi) - \frac{p}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1}^{p-1}(\xi).$$

By repeated differentiation of (3.8) we get the general formula for any order derivative. Since we are only interested in the first derivative, we ignore further details [13].

The derivatives of rational functions will clearly depend on the derivatives of their non-rational counterpart. Definition 3.6 can be generalized for NURBS.

DEFINITION 3.7. *The derivative of the i^{th} NURBS basis function is given by*

$$(3.9) \quad \frac{d}{d\xi} R_i^p(\xi) = w_i \frac{w(\xi) \frac{d}{d\xi} N_i^p(\xi) - \frac{d}{d\xi} w(\xi) N_i^p(\xi)}{(w(\xi)^2)}.$$

where w_i and $w(\xi)$ are defined in (3.4) and (3.5), respectively.

3.3. Condition number of B-Splines. In this section, we recall bounds for the condition number of B-Splines.

We need to know the bounds on B-Spline basis functions in some L_s -norm, where $s \in [1, \infty]$. We estimate the size of the coefficients of a polynomial of degree p in two dimensions when it is represented using the tensor product structure of B-Spline basis functions. The condition number of a basis can be defined as follows.

DEFINITION 3.8. *A basis $\{N_i\}$ of a normed linear space is said to be stable with respect to a vector norm if there are constants K_1 and K_2 such that for all coefficients $\{v_i\}$ the following relation holds:*

$$(3.10) \quad K_1^{-1} \|\{v_i\}\| \leq \left\| \sum_i v_i N_i \right\| \leq K_2 \|\{v_i\}\|.$$

The number $\kappa = K_1 K_2$, with K_1 and K_2 as small as possible, is called the condition number of $\{N_i\}$ with respect to $\|\cdot\|$. Note that we use the symbols $\|\cdot\|$ and $\|\{\cdot\}\|$ for the norms in the vector space and the vector norm, respectively.

Such condition numbers give an upper bound for magnification of the error in the coefficients to the function values. Indeed, if $f = \sum_i f_i N_i \neq 0$ and $g = \sum_i g_i N_i$, then it follows immediately from (3.10) that

$$\frac{\|f - g\|}{\|f\|} \leq \kappa \frac{\|\{f_i - g_i\}\|}{\|\{f_i\}\|}.$$

More details on the approximation properties and the stability of B-Splines can be found in [24, 25, 26, 30, 31, 36, 38]. We use these estimates of κ to estimate the bounds of the condition number of the stiffness matrix and the mass matrix.

It is of central importance for working with B-Spline basis functions that its condition number is bounded independently of the underlying knot sequence. That is,

the condition number of B-Splines does not depend on the multiplicity of the knots of knot vector [10, 11, 12, 15]. In [11] is a direct estimate that the worst condition number of a B-Spline of degree p with respect to any L_s -norm is bounded above by $p9^p$. It is also conjectured that the real value of κ grows like 2^p , which is superior to the direct estimate:

$$(3.11a) \quad \kappa < p9^p \quad (\text{direct estimate}),$$

$$(3.11b) \quad \kappa \sim 2^p \quad (\text{de Boor's conjecture}).$$

In [14], the exact condition number of a B-Spline basis is shown to be difficult to determine.

Scherer and Shadrin [44] proved that the upper bound of the condition number κ of a B-Spline of degree p with respect to L_s -norm is bounded by

$$(3.12) \quad \kappa < p^{\frac{1}{2}} 4^p,$$

which is closer to the conjecture in (3.11b). Scherer and Shadrin [45] proved the following result.

LEMMA 3.9. *For all p and all $s \in [1, \infty]$,*

$$(3.13) \quad \kappa < p2^p.$$

Lemma 3.9 confirms the conjecture (3.11b) up to a polynomial factor. Possible approaches to eliminate the polynomial factor are also discussed in [45]. Lemma 3.9 can be easily generalized to d -dimensions.

LEMMA 3.10. *Using a tensor product B-Spline basis of degree p in d -dimensions and (3.13), the following is immediate:*

$$(3.14) \quad \kappa < (p2^p)^d.$$

4. Estimates of condition number. In this section, we give estimates for the condition number of the stiffness matrix (in Section 4.1) and the mass matrix (in Section 4.2) obtained from isogeometric discretization. In each case, we have bounds on the condition number with respect to both h - and p -refinements. For h -refinement, upper bounds for the maximum eigenvalues, a lower bound for the minimum eigenvalue, and an upper bound for the condition number are given. For p -refinement, we prove upper and lower bounds for the maximum eigenvalue, lower bounds for the minimum eigenvalue, and upper bounds for the condition number.

4.1. Stiffness matrix. In this section, we give estimates for the condition number of the stiffness matrix with estimates for h -refinement in Section 4.1.1 and for p -refinement in Section 4.1.2.

4.1.1. h -refinement. Without loss of generality, we begin with a two-dimensional open parametric domain $\Omega = (0, 1)^2$ that we refer to as a *patch*. Given two open knot vectors $\Xi_1 = \{0 = \xi_1, \xi_2, \xi_3, \dots, \xi_{m_1} = 1\}$ and $\Xi_2 = \{0 = \eta_1, \eta_2, \eta_3, \dots, \eta_{m_2} = 1\}$, we partition the patch Ω into a mesh

$$\mathcal{Q}_h = \{Q = (\xi_i, \xi_{i+1}) \otimes (\eta_j, \eta_{j+1}), i = p_1+1, 2, \dots, m_1-p_1-1, j = p_2+1, 2, \dots, m_2-p_2-1\},$$

where Q is a two-dimensional open knot-span whose diameter is denoted by h_Q . We consider a family of quasi-uniform meshes $\{\mathcal{Q}_h\}_h$ on Ω , where $h = \max\{h_Q | Q \in \mathcal{Q}_h\}$

denotes the family index [5]. Let \mathcal{S}_h denote the B-spline space associated with the mesh \mathcal{Q}_h . Given two adjacent elements Q_1 and Q_2 , we denote by $m_{Q_1 Q_2}$ the number of continuous derivatives across their common face $\partial Q_1 \cap \partial Q_2$. In the analysis, we will use the usual Sobolev space of order $m \in \mathbb{N}$,

$$(4.1) \quad \mathcal{H}^m(\Omega) = \left\{ v \in L^2(\Omega) \text{ such that } v|_Q \in H^m(Q), \forall Q \in \mathcal{Q}_h, \text{ and } \right. \\ \left. \nabla^i(v|_{Q_1}) = \nabla^i(v|_{Q_2}) \text{ on } \partial Q_1 \cap \partial Q_2, \right. \\ \left. \forall i \in \mathbb{N} \text{ with } 0 \leq i \leq \min\{m_{Q_1 Q_2}, m-1\}, \forall Q_1, Q_2 \text{ with } \partial Q_1 \cap \partial Q_2 \neq \emptyset \right\},$$

where ∇^i has the usual meaning of i^{th} -order partial derivative. The space \mathcal{H}^m is equipped with the following semi-norms and norm

$$|v|_{\mathcal{H}^i(\Omega)}^2 = \sum_{Q \in \mathcal{Q}_h} |v|_{H^i(Q)}^2, \quad 0 \leq i \leq m, \quad \text{and} \quad \|v\|_{\mathcal{H}^m(\Omega)}^2 = \sum_{i=0}^m |v|_{\mathcal{H}^i(\Omega)}^2.$$

On a regular mesh of size h , the condition number of the finite element equations for a second-order elliptic boundary value problem can be obtained using *inverse estimates* [2, 7, 8]. Similar inverse estimates are of interest for the isogeometric framework using NURBS basis functions.

To keep the article self-contained, we recall some results from [5, 47].

THEOREM 4.1. *Let \mathcal{S}_h be the spline space consisting of piecewise polynomials of degree p associated with uniform partitions. Then there exists a constant $C = C(\text{shape})$, such that for all $0 \leq l \leq m$,*

$$(4.2) \quad \|v\|_{\mathcal{H}^m(\Omega)} \leq Ch^{l-m} \|v\|_{\mathcal{H}^l(\Omega)}, \quad \forall v \in \mathcal{S}_h.$$

The proof of the above theorem, for a particular case $m = 2$ and $l = 1$, is given in [5]. More general inverse inequalities can be easily derived following the same approach. By taking $m = 1$ and $l = 0$, the following can be easily derived from (4.2)

$$(4.3) \quad a(v, v) = \int_{\Omega} |\nabla v|^2 \leq Ch^{-2} \|v\|^2.$$

Under suitable conditions the condition number related to elliptic problems in finite element analysis scales as h^{-2} [18, 29, 48]. We prove the similar result for the stiffness matrix arising in isogeometric discretization.

We first prove

LEMMA 4.2. *There exist constants C_1 and C_2 independent of h (but may depend on p), such that for all $v = \sum_{i=1}^{n_h} v_i N_i \in \mathcal{S}_h$,*

$$(4.4) \quad C_1 h^2 \|\{v_i\}\|^2 \leq \left\| \sum_{i=1}^{n_h} v_i N_i \right\|^2 \leq C_2 h^2 \|\{v_i\}\|^2.$$

Proof. We only consider the non-trivial case: there exists some i for which $v_i \neq 0$. For any $Q \in \mathcal{Q}_h$, there are $(p+1)^2$ basis functions with non-zero support. Let $\mathcal{I}_h^Q \equiv \{i_1^Q, i_2^Q, \dots, i_{p+1}^Q\} \times \{j_1^Q, j_2^Q, \dots, j_{p+1}^Q\} \subset \{1, 2, \dots, n_h\}$ denote the index set for

the basis functions that have non-zero support in Q . Also, let $\bar{v}_q = \max_{i \in \mathcal{I}_h^Q} |v_i|$ and $\bar{v} = \max_{i=1,2,\dots,n_h} |v_i|$. Now using positivity and partition of unity properties of basis functions, the right hand side inequality can be proved as follows:

$$\begin{aligned} \|v\|^2 &= \sum_{Q \in \mathcal{Q}_h} \int_Q v^2 = \sum_{Q \in \mathcal{Q}_h} \int_Q \left(\sum_{i \in \mathcal{I}_h^Q} v_i N_i \right)^2 \leq \sum_{Q \in \mathcal{Q}_h} \int_Q \left(\bar{v}_q \sum_{i \in \mathcal{I}_h^Q} N_i \right)^2 \\ &\leq \sum_{Q \in \mathcal{Q}_h} \int_Q \bar{v}_q^2 \leq \sum_{Q \in \mathcal{Q}_h} h_Q^2 \bar{v}_q^2 \leq \sum_{Q \in \mathcal{Q}_h} h_Q^2 \sum_{i \in \mathcal{I}_h^Q} v_i^2 \\ &\leq h^2 \sum_{Q \in \mathcal{Q}_h} \sum_{i \in \mathcal{I}_h^Q} v_i^2 \leq C_2 h^2 \sum_{i=1}^{n_h} v_i^2 = C_2 h^2 \|\{v_i\}\|^2. \end{aligned}$$

For the left hand side inequality,

$$\begin{aligned} h^2 \|\{v_i\}\|^2 &= h^2 \sum_{i=1}^{n_h} v_i^2 \leq h^2 \sum_{i=1}^{n_h} \bar{v}^2 = h^2 n_h \bar{v}^2 \leq h^2 \left(\frac{C}{h} \right)^2 \bar{v}^2 = C^2 \bar{v}^2 \\ &= C^2 \|\{v_i\}\|_{L^\infty}^2 \leq C^2 K_1^2 \|v\|_{L^\infty}^2 \quad \left(\text{using (3.10), } K_1^{-1} \|\{v_i\}\|_{L^\infty} \leq \left\| \sum v_i N_i \right\|_{L^\infty} \right) \\ &\leq C^2 K_1^2 \|v\|^2. \end{aligned}$$

The result then follows by taking $C_1 = \left(\frac{1}{C^2 K_1^2} \right)$. \square

We now turn to the problem of obtaining bounds on the extremal eigenvalues and the condition number.

THEOREM 4.3. *Let A be the stiffness matrix $A = (a_{ij})$, where $a_{ij} = a(N_i, N_j) = \int_\Omega \nabla N_i \cdot \nabla N_j$. Then the upper bound on λ_{max} and lower bound on λ_{min} are given by*

$$\lambda_{max} \leq c_1 \quad \text{and} \quad \lambda_{min} \geq c_2 h^2,$$

where c_1, c_2 are constants independent of h . The bound on $\kappa(A)$ is given by

$$\kappa(A) \leq C h^{-2},$$

where C is a constant independent of h .

Proof. Let $v = \sum_{i=1}^{n_h} v_i N_i$. Then $a(v, v) = \{v_i\} \cdot A \{v_i\}$, where $\{v_i\} = \{v_1, v_2, \dots, v_{n_h}\}$.

Using the inverse estimate (4.3),

$$\frac{\{v_i\} \cdot A \{v_i\}}{\|\{v_i\}\|^2} = \frac{a(v, v)}{\|\{v_i\}\|^2} \leq \frac{C h^{-2} \|v\|^2}{\|\{v_i\}\|^2}.$$

Using (4.4),

$$\frac{C h^{-2} \|v\|^2}{\|\{v_i\}\|^2} \leq \frac{C h^{-2} C_2 h^2 \|\{v_i\}\|^2}{\|\{v_i\}\|^2} = C C_2 = c_1.$$

Hence,

$$(4.5) \quad \lambda_{max} = \sup_{v \neq 0} \frac{\{v_i\} \cdot A \{v_i\}}{\|\{v_i\}\|^2} \leq c_1.$$

On the other hand, for the bounds on λ_{\min} , by using coercivity of bilinear form $a(v, v)$,

$$\frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} = \frac{a(v, v)}{\|\{v_i\}\|^2} \geq \frac{\alpha \|v\|_{H^1}^2}{\|\{v_i\}\|^2} \geq \frac{\alpha \|v\|^2}{\|\{v_i\}\|^2}.$$

Using (4.4) again,

$$\frac{\alpha \|v\|^2}{\|\{v_i\}\|^2} \geq \frac{\alpha_1 C_1 h^2 \|\{v_i\}\|^2}{\|\{v_i\}\|^2} = \alpha_1 C_1 h^2 = c_2 h^2.$$

Hence,

$$(4.6) \quad \lambda_{\min} = \inf_{v \neq 0} \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} \geq c_2 h^2.$$

The condition number of the stiffness matrix is given by

$$\kappa(A) = \frac{\lambda_{\max}}{\lambda_{\min}}, \text{ where } \lambda_{\max} = \max_{v \neq 0} \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2}, \text{ and } \lambda_{\min} = \min_{v \neq 0} \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2}.$$

From (4.5) and (4.6),

$$(4.7) \quad \kappa(A) \leq C h^{-2}.$$

□

4.1.2. p -refinement. In this section for p -refinement, we prove upper and lower bounds for the maximum eigenvalue, lower bounds for the minimum eigenvalue, and upper bounds for the condition number.

Let \mathcal{S}_p be the tensor product space of spline functions of degree p .

The following lemma is well known generalization of a theorem of Markov due to Hill, Szechuan and Tamarkin [6, 37].

LEMMA 4.4 (Schmidt's inequality). *There exists a constant C independent of p such that for any polynomial $f(x)$ of degree p ,*

$$(4.8) \quad \int_{-1}^1 (f'(x))^2 dx \leq C p^4 \int_{-1}^1 (f(x))^2 dx.$$

Note: No such constant C exists so that (4.8) holds for all $f(x)$ with the exponent smaller than 4.

Let $I = (-1, 1)$. Using (4.8),

$$(4.9) \quad \int_I \left(\frac{dN_p(\xi)}{d\xi} \right)^2 d\xi \leq C p^4 \int_I (N_p(\xi))^2 d\xi.$$

Using (4.9),

$$(4.10) \quad \begin{aligned} \int_{\Omega} \nabla N_p(\xi, \eta) \cdot \nabla N_p(\xi, \eta) d\xi d\eta &= \int_I \int_I \left[\left(\frac{\partial N_p(\xi, \eta)}{\partial \xi} \right)^2 + \left(\frac{\partial N_p(\xi, \eta)}{\partial \eta} \right)^2 \right] d\xi d\eta \\ &\leq C p^4 \int_{I \times I} (N_p(\xi, \eta))^2 d\xi d\eta. \end{aligned}$$

Moreover, the following estimate directly follows from Schmidt's inequality and (4.10):

$$(4.11) \quad a(v, v) = \int_{\Omega} |\nabla v|^2 \leq Cp^4 \|v\|^2.$$

We now have a similar result like Lemma 4.2 for the p -refinement.

LEMMA 4.5. *There exist constants C_1 and C_2 independent of p such that for all $v = \sum_{i=1}^{n_p} v_i N_i \in \mathcal{S}_p$,*

$$(4.12) \quad \frac{C_1}{(p^2 4^p)^2} \|\{v_i\}\|^2 \leq \left\| \sum_{i=1}^{n_p} v_i N_i \right\|^2 \leq C_2 \|\{v_i\}\|^2.$$

Proof. From the stability of B-Splines there exists a constant γ that depends on the degree p such that

$$(4.13) \quad \left\| \sum_{i=1}^{n_p} v_i N_i \right\| \leq \|\{v_i\}\| \leq \gamma \left\| \sum_{i=1}^{n_p} v_i N_i \right\|,$$

From (3.14), $\gamma = p^2 4^p$. In the estimate (4.12), the right hand side inequality follows easily from nonnegativity and the partition of unity properties of basis functions. The left hand side inequality follows from (4.13). \square

For the p -refinement of isogeometric discretization, the analog to Theorem 4.3 is

THEOREM 4.6. *Let $\{N_i\}$ be a set of basis functions of \mathcal{S}_p on a unit square. Then $\kappa(A) \leq Cp^8 16^p$.*

Proof. We prove this theorem following the same approach as for the h -refinement estimates. Let $v = \sum_{i=1}^{n_p} v_i N_i$, where $\{v_i\} = \{v_1, v_2, \dots, v_{n_p}\}$. Now using (4.11) and (4.12),

$$\frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} = \frac{a(v, v)}{\|\{v_i\}\|^2} \leq \frac{Cp^4 \|v\|^2}{\|\{v_i\}\|^2} \leq \frac{Cp^4 C_2 \|\{v_i\}\|^2}{\|\{v_i\}\|^2} = CC_2 p^4 = Cp^4.$$

Hence,

$$(4.14) \quad \lambda_{\max} = \max_{v \neq 0} \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} \leq Cp^4.$$

To prove the lower bound for λ_{\min} we use (4.12) and coercivity of bilinear form,

$$\begin{aligned} \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} &= \frac{a(v, v)}{\|\{v_i\}\|^2} \geq \frac{\alpha \|v\|_{H^1}^2}{\|\{v_i\}\|^2} \geq \frac{\alpha \|v\|^2}{\|\{v_i\}\|^2} \\ &\geq \frac{\alpha \frac{C_1}{(p^2 4^p)^2} \|\{v_i\}\|^2}{\|\{v_i\}\|^2} = \frac{\alpha C_1}{(p^2 4^p)^2} = \frac{C}{(p^4 16^p)}. \end{aligned}$$

Hence,

$$(4.15) \quad \lambda_{\min} = \min_{v \neq 0} \frac{\{v_i\} \cdot A\{v_i\}}{\|\{v_i\}\|^2} \geq \frac{C}{(p^4 16^p)}.$$

From (4.14) and (4.15),

$$\kappa(A) = \frac{\lambda_{\max}}{\lambda_{\min}} \leq \frac{Cp^4}{\left(\frac{C}{(p^4 16^p)}\right)} \leq C(p^8 16^p).$$

□

REMARK 4.7. *Theorem 4.6 can be easily generalized for higher dimensions. The bound for the condition number of the stiffness matrix for a d -dimensional problem is given by $(p^{4+2d} 4^{pd})$.*

While we proved an upper bound on the maximum eigenvalue of the stiffness matrix using B-Spline basis functions, Theorem 4.6 is independent of the choice of the basis functions (it holds for all kind of basis functions, not just spline functions). From numerical experiments using B-Spline basis functions (see Table 1), we observe that λ_{\max} depends linearly on the polynomial degree p , which motivates further investigations.

The lower bound on the minimum eigenvalue depends on the stability of the B-Spline basis functions, which cannot be improved further (especially beyond the de Boor's conjecture). On the other hand, the upper bound on the maximum eigenvalue directly depends on the upper bound of the bilinear form $a(v, v)$. We can improve the bound for $a(v, v)$ given in (4.11). In the following theorem we improve this bound and provide our main result.

THEOREM 4.8. *For the two dimensional problem the improved upper bound for the condition number of the stiffness matrix A is given by*

$$(4.16) \quad \kappa(A) \leq Cp^2(p^2 4^p)^2 = Cp^6 16^p.$$

The bound for a d -dimensional problem is given by

$$(4.17) \quad \kappa(A) \leq Cp^{2d+2} 4^{pd}.$$

For the sake of clarity we will give the proof of Theorem 4.8 in parts in Lemmas 4.9-4.13.

TABLE 1
Maximum eigenvalue of the stiffness matrix A

$p \backslash h^{-1}$	1	2	4	8	16	32	64	128
2	0.36	1.42	1.42	1.49	1.50	1.50	1.50	1.50
3	0.45	1.04	1.37	1.52	1.56	1.57	1.57	1.57
4	0.41	0.94	1.33	1.72	1.81	1.83	1.84	1.84
5	0.35	0.88	1.32	1.93	2.10	2.14	2.14	2.14
6	0.34	0.85	1.32	2.12	2.40	2.46	2.47	2.47
7	0.33	0.84	1.32	2.26	2.70	2.78	2.80	2.80
8	0.32	0.83	1.33	2.36	2.99	3.11	3.13	3.14
9	0.31	0.82	1.33	2.43	3.29	3.44	3.47	3.47
10	0.31	0.82	1.34	2.47	3.56	3.77	3.80	3.81
20	0.29	0.78	1.36	2.65	5.02	6.95	7.20	7.23
30	0.29	0.78	1.36	2.69	5.28	9.38	10.55	10.66

It is clear from Table 1 that the maximum eigenvalue of the stiffness matrix is independent of p for the coarsest mesh size $h = 1$, and linearly dependent of p asymptotically. In the analysis, we consider two dimensional problem on the coarsest mesh first and extend it to finer meshes later. On the coarsest mesh we have B-Spline basis functions of degree p in one variable ξ ,

$$N_{i,\xi}^p = (-1)^i \binom{p}{i} (\xi - 1)^{p-i} \xi^i, \quad i = 0, 1, 2, \dots, p.$$

Similarly in variable η ,

$$N_{j,\eta}^p = (-1)^j \binom{p}{j} (\eta - 1)^{p-j} \eta^j, \quad j = 0, 1, 2, \dots, p.$$

Two variable B-Spline basis functions on the coarsest mesh is given by the tensor product

$$N_{i,j,\xi,\eta}^{p,p} = (-1)^{i+j} \binom{p}{i} \binom{p}{j} \xi^i \eta^j (\xi - 1)^{p-i} (\eta - 1)^{p-j}, \quad i, j = 0, 1, 2, \dots, p.$$

We construct an upper bound of the diagonal entries of the stiffness matrix on the coarsest mess i.e. single element stiffness matrix A^e .

LEMMA 4.9. *There exists a constant C independent of p , such that*

$$(4.18) \quad A_{(i,j),(i,j)}^e = a(N_{i,j,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p}) = \int_0^1 \int_0^1 \nabla N_{i,j,\xi,\eta}^{p,p} \cdot \nabla N_{i,j,\xi,\eta}^{p,p} d\xi d\eta \leq C.$$

Proof. We provide the major points of the proof. Some of the details can be found in the research report [21]. For all $i, j = 0, 1, 2, \dots, p$,

$$\begin{aligned} a(N_{i,j,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p}) &= \int_0^1 \int_0^1 \nabla N_{i,j,\xi,\eta}^{p,p} \cdot \nabla N_{i,j,\xi,\eta}^{p,p} d\xi d\eta \\ &= \binom{p}{i}^2 \binom{p}{j}^2 \int_0^1 \int_0^1 \{ i \xi^{i-1} \eta^j (\xi - 1)^{p-i} (\eta - 1)^{p-j} + \\ &\quad (p-i) \xi^i \eta^j (\xi - 1)^{p-i-1} (\eta - 1)^{p-j} \}^2 d\xi d\eta \\ &+ \binom{p}{i}^2 \binom{p}{j}^2 \int_0^1 \int_0^1 \{ j \xi^i \eta^{j-1} (\xi - 1)^{p-i} (\eta - 1)^{p-j} + \\ &\quad (p-j) \xi^i \eta^j (\xi - 1)^{p-i} (\eta - 1)^{p-j-1} \}^2 d\xi d\eta \\ &\equiv I + II. \end{aligned}$$

Now,

$$\begin{aligned} I &= \binom{p}{i}^2 \binom{p}{j}^2 \int_0^1 \int_0^1 \left(i^2 \xi^{2(i-1)} \eta^{2j} (\xi - 1)^{2(p-i)} (\eta - 1)^{2(p-j)} \right) d\xi d\eta + \\ &\quad \binom{p}{i}^2 \binom{p}{j}^2 \int_0^1 \int_0^1 \left((p-i)^2 \xi^{2i} \eta^{2j} (\xi - 1)^{2(p-i-1)} (\eta - 1)^{2(p-j)} \right) d\xi d\eta + \\ &\quad \binom{p}{i}^2 \binom{p}{j}^2 \int_0^1 \int_0^1 \left(2i(p-i) \xi^{2i-1} \eta^{2j} (\xi - 1)^{2p-2i-1} (\eta - 1)^{2(p-j)} \right) d\xi d\eta \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

After simplifying (using results on factorial functions), we get

$$\begin{aligned} I_1 &\leq \frac{1}{2} \binom{p}{i}^2 \binom{p}{j}^2 \frac{(2i)!(2p-2i)!}{(2p-1)!} \frac{(2j)!(2p-2j)!}{(2p+1)!}, \\ I_2 &\leq \frac{1}{2} \binom{p}{i}^2 \binom{p}{j}^2 \frac{(2i)!(2p-2i)!}{(2p-1)!} \frac{(2j)!(2p-2j)!}{(2p+1)!}, \\ I_3 &= -\frac{1}{2} \binom{p}{i}^2 \binom{p}{j}^2 \frac{(2i)!(2p-2i)!}{(2p-1)!} \frac{(2j)!(2p-2j)!}{(2p+1)!}. \end{aligned}$$

For all $i = 0, 1, 2, \dots, p$,

$$\begin{aligned} I &= I_1 + I_2 + I_3 \\ &= \begin{cases} I_2, & \text{if } i = 0, \\ I_1 + I_2 + I_3, & \text{if } i = 1, 2, \dots, p-1, \\ I_1, & \text{if } i = p, \end{cases} \\ &\leq \left\{ \binom{p}{i}^2 \frac{(2i)!(2p-2i)!}{(2p)!} \right\} \left\{ \binom{p}{j}^2 \frac{(2j)!(2p-2j)!}{(2p)!} \right\} = I_a I_b, \text{ where} \\ I_a &= \binom{p}{i}^2 \frac{(2i)!(2p-2i)!}{(2p)!} = \frac{p!p!}{i!i!(p-i)!(p-i)!} \frac{(2i)!(2p-2i)!}{(2p)!}, \\ I_b &= \binom{p}{j}^2 \frac{(2j)!(2p-2j)!}{(2p)!} = \frac{p!p!}{j!j!(p-j)!(p-j)!} \frac{(2j)!(2p-2j)!}{(2p)!}. \end{aligned}$$

We prove that $I_a \leq C$ by induction on p , where C is a constant independent of p . For $p = 1$, we have $I_a = 1$ for all $i = 0, 1$. Hence, the result holds for the base case. Assume that the result holds for $p = m$ and for all $i = 0, 1, 2, \dots, m$,

$$(4.19) \quad \frac{m!m!}{i!i!(m-i)!(m-i)!} \frac{(2i)!(2m-2i)!}{(2m)!} \leq C.$$

Now we show that the result holds for $p = m+1$ and for all $i = 0, 1, 2, \dots, m+1$. We have

$$\begin{aligned} &\frac{(m+1)!(m+1)!}{i!i!(m+1-i)!(m+1-i)!} \frac{(2i)!(2(m+1)-2i)!}{(2(m+1))!} \\ &= \begin{cases} \left(\frac{m^2+2m+1}{4m^2+6m+2} \right) \left(\frac{4(m-i)^2+6(m-i)+2}{(m-i)^2+2(m-i)+1} \right) \left\{ \frac{m!m!}{i!i!(m-i)!(m-i)!} \frac{(2i)!(2m-2i)!}{(2m)!} \right\}, \\ \quad \text{if } i = 0, 1, 2, \dots, m, \\ 1, \text{ if } i = m+1. \end{cases} \end{aligned}$$

Using (4.19) and since $\left(\frac{m^2+2m+1}{4m^2+6m+2} \right) \left(\frac{4(m-i)^2+6(m-i)+2}{(m-i)^2+2(m-i)+1} \right) \leq 1$, we get for all $i = 0, 1, 2, \dots, m+1$,

$$\frac{(m+1)!(m+1)!}{i!i!(m+1-i)!(m+1-i)!} \frac{(2i)!(2(m+1)-2i)!}{(2(m+1))!} \leq C.$$

We now have $I_a \leq C$, where C is a constant independent of p . Similarly we can obtain that $I_b \leq C$. Hence,

$$I = I_a I_b \leq C.$$

Proceeding in the same way for \mathbb{I} , we can prove that

$$\mathbb{I} \leq C.$$

Finally,

$$a(N_{i,j,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p}) = I + \mathbb{I} \leq C.$$

□

Thus, we have proved that $a(N_{i,j,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p})$ is bounded by a constant independent of p . Since the upper bound of the diagonal entries is the upper bound of all the entries of the stiffness matrix, the maximum entry of the stiffness matrix is bounded by a constant independent of p ,

$$(4.20) \quad a(N_{i,j,\xi,\eta}^{p,q}, N_{k,l,\xi,\eta}^{p,q}) \leq C.$$

Similarly, we can prove for three dimensional problem that

$$(4.21) \quad a(N_{i,j,k,\xi,\eta,\zeta}^{p,q,r}, N_{l,m,n,\xi,\eta,\zeta}^{p,q,r}) \leq C.$$

Using (4.20) and (4.21) we have

LEMMA 4.10. *The maximum eigenvalue of the element stiffness matrix A^e can be bounded below by a constant C independent of p ,*

$$\lambda_{\max}(A^e) \geq C.$$

Proof. We prove this by using the basics of matrix norms. The max-norm of a matrix is the element-wise norm defined by

$$\|A^e\|_{\max} = \max\{|a_{ij}|\}.$$

From (4.18),

$$\max\{|a_{ij}|\} = C,$$

where C is independent of p . By the equivalence of norms we have

$$\|A^e\|_2 \geq \|A^e\|_{\max} = C.$$

Hence,

$$\lambda_{\max}(A^e) \geq C.$$

□

To bound λ_{\max} from above we bound the spectral norm by the ℓ_1 -norm in

LEMMA 4.11. *For any fixed k and l such that $0 \leq k, l \leq p$ and for any $0 \leq i, j \leq p$,*

$$\sum_{i=0}^p \sum_{j=0}^p |a(N_{k,l,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p})| < C,$$

where C is a constant independent of p .

Proof. We again provide the main steps and for details refer the reader to the research report [21]. We have

$$N_{0,0,\xi,\eta}^{p,p} = (1 - \xi)^p (1 - \eta)^p.$$

We first prove

$$\sum_{i=0}^p \sum_{j=0}^p |a(N_{0,0,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p})| < C,$$

where C is a constant independent of p . We have

$$\begin{aligned} a(N_{0,0,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p}) &= \int_0^1 \int_0^1 \nabla N_{0,0,\xi,\eta}^{p,p} \cdot \nabla N_{i,j,\xi,\eta}^{p,p} d\xi d\eta \\ &= \int_0^1 \int_0^1 \left(\frac{\partial}{\partial \xi} N_{0,0,\xi,\eta}^{p,p} \frac{\partial}{\partial \xi} N_{i,j,\xi,\eta}^{p,p} \right) d\xi d\eta + \int_0^1 \int_0^1 \left(\frac{\partial}{\partial \eta} N_{0,0,\xi,\eta}^{p,p} \frac{\partial}{\partial \eta} N_{i,j,\xi,\eta}^{p,p} \right) d\xi d\eta \\ &= I + II. \end{aligned}$$

Now,

$$\begin{aligned} I &= -p \binom{p}{i} \binom{p}{j} \left(\int_0^1 \eta^j (1 - \eta)^{2p-j} d\eta \right) \\ &\quad \left(\int_0^1 i \xi^{i-1} (1 - \xi)^{2p-i-1} d\xi - \int_0^1 (p-i) \xi^i (1 - \xi)^{2p-i-2} d\xi \right) \\ I &= \begin{cases} \binom{p}{j} \frac{p^2}{(4p^2-1)} \frac{(j)!(2p-j)!}{(2p)!}, & \text{if } i = 0, \\ -p \binom{p}{i} \binom{p}{j} \frac{2p}{(2p+1)} \frac{(i)!(2p-i)!}{(2p)!} \frac{(j)!(2p-j)!}{(2p)!} \frac{1}{(2p-i)} \left(1 - \frac{(p-i)}{(2p-i-1)} \right), & \text{if } i = 1, 2, \dots, p-1, \\ -\binom{p}{j} \frac{2p}{(2p+1)} \frac{(p)!(p)!}{(2p)!} \frac{(j)!(2p-j)!}{(2p)!}, & \text{if } i = p. \end{cases} \end{aligned}$$

A similar expression can be obtained for II . We want to calculate

$$\sum_{i=0}^p \sum_{j=0}^p |a(N_{0,0,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p})|.$$

For $i = 0$,

$$\begin{aligned} \sum_{j=0}^p |I| &= \sum_{j=0}^p \binom{p}{j} \frac{p^2}{(4p^2-1)} \frac{(j)!(2p-j)!}{(2p)!} < \frac{1}{3} \sum_{j=0}^p \binom{p}{j} \frac{(j)!(2p-j)!}{(2p)!} \\ &= \frac{1}{3} \sum_{j=0}^p \frac{(p)!(2p-j)!}{(2p)!(p-j)!} < \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{p!p!}{(2p)!} \right) < 1. \end{aligned}$$

For $i = 1, 2, \dots, p-1$,

$$\begin{aligned} & \sum_{i=1}^{p-1} \sum_{j=0}^p |I| \\ &= \sum_{i=1}^{p-1} \sum_{j=0}^p \binom{p}{i} \binom{p}{j} \frac{2p^2}{(2p+1)} \frac{(i)!(2p-i)!}{(2p)!} \frac{(j)!(2p-j)!}{(2p)!} \frac{1}{(2p-i)} \left(1 - \frac{(p-i)}{(2p-i-1)}\right) \\ &< \frac{1}{2} \sum_{i=1}^{p-1} \frac{p!(2p-i-2)!}{(2p-2)!(p-i)!} < \frac{1}{2} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{p!(p-1)!}{(2p-2)!} \right) < 1. \end{aligned}$$

For $i = p$,

$$\begin{aligned} \sum_{j=0}^p |I| &= \sum_{j=0}^p \binom{p}{j} \frac{2p}{(2p+1)} \frac{(p)!(p)!}{(2p)!} \frac{(j)!(2p-j)!}{(2p)!} < \sum_{j=0}^p \frac{p!p!(2p-j)!}{(2p)!(2p)!(p-j)!} \\ &< \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{(p!)^4}{((2p)!)^2} \right) < 1. \end{aligned}$$

Hence,

$$(4.22) \quad \sum_{i=0}^p \sum_{j=0}^p |I| < C,$$

where C is independent of p . Similarly, we have

$$(4.23) \quad \sum_{i=0}^p \sum_{j=0}^p |\mathbb{I}| < C.$$

Therefore, from (4.22) and (4.23),

$$(4.24) \quad \sum_{i=0}^p \sum_{j=0}^p |a(N_{0,0,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p})| < C,$$

where C is a constant independent of p .

We have bounded by a constant the absolute row sum for the first row of the element stiffness matrix. Since on a uniform mesh the absolute row sum for all rows of the element stiffness matrix are of the same order upto a constant, we get the desired result:

$$\sum_{i=0}^p \sum_{j=0}^p |a(N_{k,l,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p})| < C,$$

for any fixed k and l such that $0 \leq k, l \leq p$ and for any $0 \leq i, j \leq p$, where C is a constant independent of p . \square

Similar results can be obtained for higher dimensions. The next lemma is a direct consequence of the Lemma 4.11 and gives an upper bound for the maximum eigenvalue.

LEMMA 4.12. *The maximum eigenvalue of the element stiffness matrix A^e can be bounded above by a constant C that is independent of p :*

$$\lambda_{\max}(A^e) \leq C.$$

Proof. We have

$$\sum_{i=0}^p \sum_{j=0}^p |a(N_{k,l,\xi,\eta}^{p,p}, N_{i,j,\xi,\eta}^{p,p})| < C,$$

where C is a constant independent of p , which implies $\|A^e\|_1 \leq C$. Since A^e is a symmetric matrix, we have $\|A^e\|_1 = \|A^e\|_\infty$. Therefore, using $\|A^e\|_2 \leq \sqrt{\|A^e\|_1 \|A^e\|_\infty}$, we get

$$\|A^e\|_2 \leq \|A^e\|_1 \leq C.$$

Thus,

$$\lambda_{\max}(A^e) \leq C.$$

□

From Lemmas 4.10 and 4.12, for the element stiffness matrix we have

LEMMA 4.13. $\lambda_{\max}(A^e) = C$, where C is a constant independent of p .

The results in Lemma 4.10 and Lemma 4.12 are proved for an element stiffness matrix on a single element mesh. Obviously these results hold for all element stiffness matrices on finer meshes. Therefore, Lemma 4.13 holds for all element stiffness matrices on refined meshes.

Now we bound the spectral norm (maximum eigenvalue) of the global stiffness matrix by its ℓ_1 -norm, i.e., the maximum of the row sum over all the rows of the global stiffness matrix. The ℓ_1 -norm of global stiffness matrix will depend on the ℓ_1 norm of the element stiffness matrices and on their assembly. Therefore, the bounds for maximum eigenvalue of the global stiffness matrix can be expressed in terms of the maximum eigenvalues of the corresponding element stiffness matrices and the maximum number of overlaps within the rows and columns of element stiffness matrices (which is the number of element stiffness matrices that contributes at a particular nonzero position in the global matrix).

In the process of assembling the global stiffness matrix, the overlaps within the element stiffness matrices depend on the regularity of the basis functions used in the discretization. For C^0 - and C^{p-1} -continuous basis function the overlaps within the elements will be minimum and maximum, respectively. It is easy to see that for C^{p-1} -continuous basis functions the overlaps will be in $(p+1)^2$ knot spans (e.g., see Fig. 1).

Using the bound for maximum eigenvalue of element stiffness matrices we state the following lemma for the bound for maximum eigenvalue of global stiffness matrix.

LEMMA 4.14. *The maximum eigenvalue of the global stiffness matrix $\lambda_{\max}(A) = Cp^2$, where C is a constant independent of p .*

Proof. In the assembly of the element stiffness matrices, the maximum number of overlaps for a particular nonzero position in the global matrix is $(p+1)^2$. We have

$$\begin{aligned} \|A\|_1 &\leq (\text{maximum number of overlaps}) \times \|A^e\|_1, \\ &\leq ((p+1)^2) \times C. \end{aligned}$$

The inequality $\|A\|_2 \leq \sqrt{\|A\|_1 \|A\|_\infty}$ and symmetry of A imply

$$\|A\|_2 \leq \|A\|_1 \leq C(p+1)^2.$$

Hence,

$$\lambda_{\max}(A) \leq Cp^2.$$

□

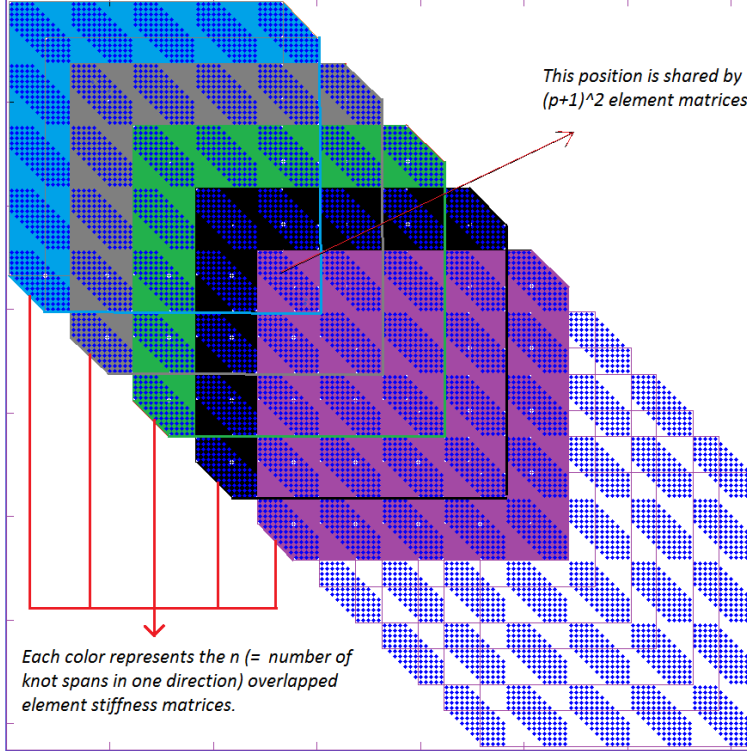
Now, using the bound for maximum eigenvalue given in Lemma 4.14, the proof of Theorem 4.8 follows directly.

REMARK 4.15. *The estimate for maximum eigenvalue given in Lemma 4.14 is not sharp. In reality this estimate is not quadratic in p . This can be explained by*

the following observation. In the overlapping all of the elements of element stiffness matrices are not summed in absolute value. Some of the negative entries overlap with positive entries which in result reduces the row sum of the global stiffness matrix. From our numerical experiments, we conjecture the following.

The maximum eigenvalue of the global stiffness matrix $\lambda_{\max}(A) = Cp$, where C is a constant independent of p .

FIG. 1. Global stiffness matrix: Assembly of element stiffness matrices for $p = 4$ on 8×8 spans



REMARK 4.16. We used the condition number of B-Splines $\kappa \sim p2^p$ and $\lambda_{\max} \sim p^2$ in reaching the above estimates. If we use the de Boor's conjecture (the condition number of B-Splines $\kappa \sim 2^p$) and $\lambda_{\max} \sim p$ (see Remark 4.15) instead, then the upper bound of the stiffness matrix can be further improved and given by

$$(4.25) \quad \kappa(A) \leq Cp4^{pd}.$$

4.2. Mass matrix. In this section, we give estimates for the condition number of the mass matrix with estimates for h -refinement in Section 4.2.1 and for p -refinement in Section 4.2.2.

4.2.1. h -refinement. Let $M = (m_{ij})$ be the mass matrix, where

$$m_{ij} = (N_i, N_j) = \int_{\Omega} N_i N_j \quad i, j = 1, 2, \dots, n_h.$$

The following lemma gives estimates for the maximum and minimum eigenvalues of the mass matrix with respect to h .

LEMMA 4.17. *For the extremal eigenvalues of the mass matrix $M = (m_{ij}) = (N_i, N_j)$,*

$$C_1 h^2 \leq \lambda_{\min} \leq \lambda_{\max} \leq C_2 h^2,$$

where C_1, C_2 are constants independent of h . Furthermore,

$$c_1 \leq \kappa(M) \leq c_2,$$

where c_1, c_2 are constants independent of h .

Proof. Using (4.4), we bound both the extremal eigenvalues of the mass matrix. For the minimum eigenvalue,

$$\frac{\{v_i\} \cdot M\{v_i\}}{\|\{v_i\}\|^2} = \frac{(v, v)}{\|\{v_i\}\|^2} \geq \frac{C_1 h^2 \|\{v\}\|^2}{\|\{v_i\}\|^2} = C_1 h^2.$$

For the maximum eigenvalue,

$$\frac{\{v_i\} \cdot M\{v_i\}}{\|\{v_i\}\|^2} = \frac{(v, v)}{\|\{v_i\}\|^2} \leq \frac{C_2 h^2 \|\{v\}\|^2}{\|\{v_i\}\|^2} = C_2 h^2.$$

So,

$$C_1 h^2 \leq \lambda_{\min} \leq \lambda_{\max} \leq C_2 h^2.$$

Hence,

$$c_1 \leq \kappa(M) \leq c_2.$$

□

4.2.2. p -refinement. In this section, we estimate the bounds on the extremal eigenvalues and the condition number of the mass matrices for p -refinement.

LEMMA 4.18. *The element mass matrix is a positive matrix and all of the entries of the element mass matrix are bounded above by $\frac{C}{(2p+1)^2}$, where C is a constant independent of p .*

Proof. We have

$$\begin{aligned} M_{(i,j),(k,l)}^e &= (N_{i,j,\xi,\eta}^{p,p}, N_{k,l,\xi,\eta}^{p,p}) = \int_0^1 \int_0^1 N_{i,j,\xi,\eta}^{p,p} \cdot N_{k,l,\xi,\eta}^{p,p} d\xi d\eta \\ &= \int_0^1 \int_0^1 \left((-1)^{i+j} \binom{p}{i} \binom{p}{j} \xi^i \eta^j (\xi-1)^{p-i} (\eta-1)^{p-j} \right) \\ &\quad \left((-1)^{k+l} \binom{p}{k} \binom{p}{l} \xi^k \eta^l (\xi-1)^{p-k} (\eta-1)^{p-l} \right) d\xi d\eta \\ &= (I)(II), \end{aligned}$$

where

$$\begin{aligned} I &= \binom{p}{i} \binom{p}{k} \left(\int_0^1 \xi^{(i+k+1)-1} (1-\xi)^{(2p-i-k+1)-1} d\xi d\eta \right) \\ &= \frac{p! p!}{i! k! (p-i)! (p-k)!} \frac{(i+k)!(2p-i-k)!}{(2p+1)!} \\ &= \frac{1}{2p+1} \left\{ \frac{p! p!}{i! k! (p-i)! (p-k)!} \frac{(i+k)!(2p-i-k)!}{(2p)!} \right\} = \frac{1}{2p+1} I_1, \end{aligned}$$

and

$$\begin{aligned}
\mathbb{I} &= \binom{p}{j} \binom{p}{l} \left(\int_0^1 \eta^{(j+l+1)-1} (1-\eta)^{(2p-j-l+1)-1} d\xi d\eta \right) \\
&= \frac{p!p!}{j!l!(p-j)!(p-l)!} \frac{(j+l)!(2p-j-l)!}{(2p+1)!} \\
&= \frac{1}{2p+1} \left\{ \frac{p!p!}{j!l!(p-j)!(p-l)!} \frac{(j+l)!(2p-j-l)!}{(2p)!} \right\} = \frac{1}{2p+1} \mathbb{I}_1.
\end{aligned}$$

By induction on p we easily obtain that (as we proved in Lemma 4.9),

$$I_1 = \left\{ \frac{p!p!}{i!k!(p-i)!(p-k)!} \frac{(i+k)!(2p-i-k)!}{(2p)!} \right\} \leq C.$$

Similarly, $\mathbb{I}_1 \leq C$. Therefore

$$(4.26) \quad M_{(i,j),(k,l)}^e \leq \frac{C}{(2p+1)^2}.$$

It is also clear that for all $p \geq 1$ and $i, k = 0, 1, 2, \dots, p$, $I_1 > 0$, and $\mathbb{I}_1 > 0$. Hence, the mass matrix $M_{(i,j),(k,l)}^e$ is a positive matrix. \square

LEMMA 4.19. *The maximum eigenvalue of the element mass matrix M^e can be bounded below by*

$$\lambda_{\max}(M^e) \geq \frac{C}{(2p+1)^2}.$$

Proof. Following the proof of Lemma 4.10 and (4.26) we get the desired result. \square

To bound λ_{\max} from above we bound the spectral norm by the ℓ_1 -norm of the mass matrix. In the following lemma we first compute the ℓ_1 -norm of the mass matrix.

LEMMA 4.20. *For the mass matrix M^e on the coarsest mesh,*

$$\|M^e\|_1 = \frac{1}{(p+1)^2}.$$

Proof. We have

$$\begin{aligned}
\|M\|_1 &= \max_{i,j} \sum_{k,l} \int_0^1 \int_0^1 N_{i,j,\xi,\eta}^{p,p} \cdot N_{k,l,\xi,\eta}^{p,p} d\xi d\eta = \max_{i,j} \sum_{k,l} (N_{i,j,\xi,\eta}^{p,p}, N_{k,l,\xi,\eta}^{p,p}) \\
&= \max_{i,j} (N_{i,j,\xi,\eta}^{p,p}, \sum_{k,l} N_{k,l,\xi,\eta}^{p,p}) = \max_{i,j} (N_{i,j,\xi,\eta}^{p,p}, 1) \quad \left(\text{since } \sum_{k,l} N_{k,l,\xi,\eta}^{p,p} = 1 \right) \\
&= \max_{i,j} \int_0^1 \int_0^1 N_{i,j,\xi,\eta}^{p,p} d\xi d\eta.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_0^1 \int_0^1 N_{i,j,\xi,\eta}^{p,p} d\xi d\eta &= \int_0^1 \int_0^1 (-1)^{i+j} \binom{p}{i} \binom{p}{j} \xi^i \eta^j (\xi-1)^{p-i} (\eta-1)^{p-j} d\xi d\eta \\
&= \binom{p}{i} \binom{p}{j} \left(\int_0^1 \xi^{(i+1)-1} (1-\xi)^{(p-i+1)-1} d\xi \right) \left(\int_0^1 \eta^{(j+1)-1} (1-\eta)^{(p-j+1)-1} d\eta \right) \\
&= \binom{p}{i} \binom{p}{j} \left(\frac{\Gamma(i+1)\Gamma(p-i+1)}{\Gamma(p+2)} \right) \left(\frac{\Gamma(j+1)\Gamma(p-j+1)}{\Gamma(p+2)} \right) \\
&= \frac{p!}{i!(p-i)!} \frac{p!}{j!(p-j)!} \frac{i!(p-i)!}{(p+1)!} \frac{j!(p-j)!}{(p+1)!} = \frac{1}{(p+1)^2}.
\end{aligned}$$

Hence,

$$\max_{i,j} \int_0^1 \int_0^1 N_{i,j,\xi,\eta}^{p,p} d\xi d\eta = \frac{1}{(p+1)^2}.$$

□

The symmetry of M^e implies

$$(4.27) \quad \|M^e\|_\infty = \|M^e\|_1 = \frac{1}{(p+1)^2}.$$

LEMMA 4.21. *The maximum eigenvalue of the mass matrix M^e on the coarsest mesh can be bounded above by*

$$\lambda_{\max}(M^e) \leq C \frac{1}{(p+1)^2}.$$

Proof. We have the following inequality for matrix norms

$$\|M^e\|_2^2 \leq \|M^e\|_1 \|M^e\|_\infty.$$

Using Lemma 4.20 and (4.27) we get the bound on the spectral norm of M^e ,

$$\|M^e\|_2 \leq C \frac{1}{(p+1)^2}.$$

□

REMARK 4.22. *In fact, for the coarsest mesh we get $\lambda_{\max}(M^e) = \frac{1}{(p+1)^2}$ by Lemma 4.18 and by [49, Lemma 2.5].*

Using the same argument as in the stiffness matrix case we can give the estimate for the maximum eigenvalue of the global mass matrix using the estimates for the element mass matrices.

LEMMA 4.23. *The maximum eigenvalue of the global mass matrix M can be bounded above by*

$$\lambda_{\max}(M) \leq C,$$

where C is a constant independent of p (may depend on h).

Proof. Following the proof of Lemma 4.14, we have

$$\lambda_{\max}(M) \leq C((p+1)^2) \times \frac{1}{(p+1)^2}.$$

Hence,

$$\lambda_{\max}(M) \leq C.$$

□

REMARK 4.24. *Unlike the stiffness matrix case, this estimate for the mass matrix case is sharp. Since all of the entries of the mass matrix are positive, therefore in the overlapping, the entries of element mass matrices are always added up without any cancellations or reductions.*

LEMMA 4.25. *There exists a constant C that is independent of p such that the minimum eigenvalue of the mass matrix M can be bounded below by*

$$\lambda_{\min}(M) \geq \frac{C}{p^4 16^p}.$$

Proof. To bound the minimum eigenvalue from below we use the left hand side inequality of (4.12):

$$\frac{\{v_i\} \cdot M\{v_i\}}{\|\{v_i\}\|^2} = \frac{(v, v)}{\|\{v_i\}\|^2} \geq \frac{\frac{C}{p^4 16^p} \|\{v_i\}\|^2}{\|\{v_i\}\|^2} = \frac{C}{p^4 16^p}.$$

Therefore, $\lambda_{\min}(M) \geq \frac{C}{p^4 16^p}$, where C is a constant that is independent of p . □

The following lemma gives us the upper bound for the condition number of the mass matrix.

LEMMA 4.26. *The condition number of the mass matrix M is bounded above by*

$$\kappa(M) \leq Cp^4 16^p,$$

where C is a constant that is independent of p .

Proof. From Lemma 4.21 and Lemma 4.25,

$$\frac{C}{p^4 16^p} \leq \lambda_{\min} \leq \lambda_{\max} \leq C.$$

Hence,

$$\kappa(M) \leq Cp^4 16^p.$$

□

REMARK 4.27. *The above bound can be easily generalized for a d -dimensional problem:*

$$(4.28) \quad \kappa(M) \leq p^{2d} 4^{pd}.$$

Following Remark 4.16 and using de Boor's conjecture (3.11b), the upper bound for the condition number of the mass matrix can be further improved and given by

$$(4.29) \quad \kappa(M) \leq 4^{pd}.$$

REMARK 4.28. *We have done all the analysis on the parametric domain $(0, 1)^2$. To get the results for the physical domain we can define an invertible NURBS geometrical map from the parametric domain to the physical domain. With suitable transformations we get the results for the physical domain. For details, see [5].*

5. Numerical results. In this section, we provide the numerical results for h -refined (in Section 5.1) and p -refined (in Section 5.2) stiffness and mass matrices. The numerical discretizations are performed using the Matlab toolbox GeoPDEs [16, 17].

5.1. h -refinement. For h -refinement, the condition number of the stiffness matrix is shown in Table 2. Numerical results are provided from $p = 2$ to $p = 5$. In the classical finite element method, the condition number of the stiffness matrix is of order h^{-2} even for a coarse mesh-size. However, in isogeometric discretizations, for higher p on coarse mesh, the condition number is highly influenced by the stability constant of B-Splines. The condition number of B-Splines heavily depends on the polynomial degree (see Section 3) and scales as $(p2^p)^d$. The factor $(p2^p)^d$ dominates the factor h^{-2} for coarse meshes. Nevertheless, the numerical results support the theoretical findings asymptotically (for reasonably refined meshes) for any polynomial degree.

In Table 3, we present the condition number of the mass matrix. We see that the condition number is bounded uniformly by a constant independent of h , which confirms the theoretical estimates.

5.2. p -refinement. We perform numerical experiments for p -refinement to obtain the maximum and minimum eigenvalues, and the condition number of the stiffness matrix and the mass matrix. The eigenvalues and the condition number are obtained on the coarsest mesh and the finest mesh. For higher p ($p > 10$) roundoff errors start contaminating the results and we stop reporting with 10.

In Tables 4 and 5, we present the extremal eigenvalues and the condition number of the stiffness matrix for $p = 2$ to $p = 10$. We observe that the maximum eigenvalue scales as a constant independent of p for the coarsest mesh and linearly dependent on p for refined meshes, and that the minimum eigenvalue is bounded from below by the bound given in Theorem 4.6.

The extremal eigenvalues and the condition number of the mass matrix for $p = 2$ to $p = 10$ are presented in Table 6 and Table 7. Numerical results confirm the theoretical estimates given in Lemma 4.21, Lemma 4.23, Lemma 4.25, and Lemma

TABLE 2
Condition number of the stiffness matrix A

$p \backslash h^{-1}$	2	4	8	16	32	64	128
2	4.00	4.00	5.22	19.77	78.14	311.58	1245.36
3	30.93	29.51	29.19	28.56	82.10	327.21	1307.67
4	339.92	269.23	240.03	222.55	215.00	381.73	1525.40
5	4177.20	3220.60	2148.25	1812.58	1700.63	1688.11	1781.51

TABLE 3
Condition number of the mass matrix M

$p \backslash h^{-1}$	2	4	8	16	32	64	128
2	89.679	109.68	108.51	109.85	111.29	111.69	111.79
3	915.558	799.941	737.379	708.010	715.89	719.45	720.33
4	11773.17	6795.46	5381.96	4762.53	4750.07	4779.41	4786.90
5	163371.70	77448.11	42580.04	33560.40	32587.27	32808.69	32871.70

4.26.

TABLE 4
 λ_{\max} , λ_{\min} , and $\kappa(A)$ on the coarsest mesh

p	λ_{\max}	λ_{\min}	$\kappa(A)$
2	0.35	3.5e-01	1.0e+00
3	0.45	3.8e-02	1.1e+01
4	0.41	2.9e-03	1.3e+02
5	0.35	2.1e-04	1.6e+03
6	0.33	1.5e-05	2.1e+04
7	0.33	1.1e-06	2.9e+05
8	0.31	7.8e-08	4.0e+06
9	0.30	5.4e-09	5.6e+07
10	0.30	3.7e-10	8.1e+08

6. Conclusions. We have provided the bounds for the minimum eigenvalue, maximum eigenvalue, and the condition numbers of the stiffness and mass matrices for the Laplace operator with h - and p -refinements of the isogeometric discretizations that are based on B-Spline (NURBS) basis functions. We proved that in the h -refinement case, like the classical finite element method, the condition number of the stiffness matrix scales as h^{-2} . For the mass matrix, it scales as constant independent of h . For the p -refinement case, we proved that the condition number of the stiffness and mass matrices grow exponentially in p .

The estimates for the minimum eigenvalues of the stiffness and mass matrices depend on the stability constant of B-Splines. In reaching these estimates we have used the stability constant of B-Splines as $p2^p$. Using the de Boor's conjecture (the stability constant of B-Splines given by 2^p , which is the best known bound), these estimates can be further improved according to Remarks 4.16 and 4.27.

Unfortunately, a sharp estimate for the stability constant is unknown. Therefore, a sharp estimate for the minimum eigenvalue cannot be determined at this time and will be the subject of future research by us and others. It is a very difficult problem.

Acknowledgments. The authors would like to thank Prof. U. Langer (Johannes Kepler University Linz, Austria) and Prof. L. Zikatanov (Pennsylvania State University, USA) for helpful suggestions on the topic of this paper.

REFERENCES

- [1] F. Auricchio, L. Beirao da Veiga, A. Buffa, C. Lovadina, A. Reali and G. Sangalli. A fully "locking-free" isogeometric approach for plane linear elasticity problems: A stream function formulation. *Comput. Methods Appl. Mech. Engrg.* **197**, 160-172, 2007.
- [2] O. Axelsson and V. A. Barker. *Finite Element Solution of Boundary Value Problems: Theory and Computation*. Society for Industrial and Applied Mathematics Philadelphia, PA, USA, 2001.
- [3] I. Babuska, J.E. Osborn. Finite element-Galerkin approximation of the eigenvalues and eigenvectors of selfadjoint problems *Mathematics of computation*, Volume 52, No. 186. 275-297, 1989.
- [4] I. Babuska, B.A. Szabo, and I.N. Katz. The p -version of the Finite Element Method. *SIAM Journal on Numerical Analysis*, Volume 18, No. 3. 515-545, 1981.
- [5] Y. Bazilevs, L. Beirao Da Veiga, J.A. Cottrell, T.J.R. Hughes and G. Sangalli. Isogeometric analysis: Approximation, Stability and error estimates for h -refined meshes. *Math. Models Methods Appl. Sci.* **16(7)**, 1031-1090, 2006.

TABLE 5
 λ_{\max} , λ_{\min} , and $\kappa(A)$ on the finest mesh

p	λ_{\max}	λ_{\min}	$\kappa(A)$
2	1.50	1.2e-03	1.2e+03
3	1.58	1.2e-03	1.3e+03
4	1.84	1.2e-03	1.5e+03
5	2.14	1.2e-03	1.8e+03
6	2.47	1.8e-04	1.4e+04
7	2.80	2.6e-05	1.1e+05
8	3.14	3.6e-06	8.8e+05
9	3.47	4.9e-07	7.1e+06
10	3.81	6.6e-08	5.7e+07

- [6] R. Bellman. A note on an inequality of E. Schmidt. *Amer. Math. Soc.*, **50**, 734-736, 1944.
- [7] D. Braess. *Finite Elements: Theory, Fast Solvers and Applications in Solid Mechanics*. Cambridge University Press, 2007.
- [8] P.G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North Holland Publishing Company, 1978.
- [9] J.A. Cottrell, T.J.R. Hughes and Y. Bazilevs. *Isogeometric Analysis: Toward Integration of CAD and FEA*. Wiley, 2009.
- [10] C. de Boor. On Calculating with B-Splines. *Journal of Approximation Theory* **6**, pp. 50-62, 1972.
- [11] C. de Boor. On local linear functionals which vanish at all B-splines but one. *Theory of Approximation with Applications*, A. G. Law and N. B. Sahney (eds.), Academic Press (New York). 120-145, 1976.
- [12] C. de Boor. Splines as linear combinations of B-Splines, a survey. In *Approximation Theory II*, G. G. Lorentz, C. K. Chui and L. L. Schumaker, (Eds.), Academic Press (New York), pp. 1-47, 1976.
- [13] C. de Boor. *A Practical Guide to Splines*. Springer-Verlag, New York, 1978.
- [14] C. de Boor. The exact condition of the B-Spline basis may be hard to determine. *Journal of Approximation theory*. **60**, 344-359, 1990.
- [15] C. de Boor and J.W. Daniel. Splines with Nonnegative B-Spline Coefficients. *Mathematics of Computation* **28(126)**, pp. 565-568, 1974.
- [16] C. de Falco, A. Reali and R. Vazquez. GeoPDEs: A research tool for Isogeometric Analysis of PDEs. *Adv. Eng. Softw.* **42**, 1020-1034, 2011.
- [17] C. de Falco, A. Reali and R. Vazquez. GeoPDEs webpage. <http://geopdes.sourceforge.net>
- [18] A. Ern, J.-L. Guermond. Evaluation of the condition number in linear systems arising in finite element approximations. *ESAIM Math. Mod. Numer. Anal.* **40(1)**, 29-48, 2006.

TABLE 6
 λ_{\max} , λ_{\min} , and $\kappa(M)$ on the coarsest mesh

p	λ_{\max}	λ_{\min}	$\kappa(M)$
2	1.1e-01	1.1e-03	1.0e+02
3	6.2e-02	5.1e-05	1.2e+03
4	4.0e-02	2.5e-06	1.5e+04
5	2.7e-02	1.3e-07	2.1e+05
6	2.0e-02	6.9e-09	2.9e+06
7	1.5e-02	3.7e-10	4.1e+07
8	1.2e-02	2.0e-11	5.9e+08
9	1.0e-02	1.1e-12	8.5e+09
10	8.2e-03	6.6e-14	1.2e+11

TABLE 7
 λ_{\max} , λ_{\min} , and $\kappa(M)$ on the finest mesh

p	λ_{\max}	λ_{\min}	$\kappa(M)$
2	6.1e-05	5.5e-07	1.1e+02
3	6.1e-05	8.5e-08	7.2e+02
4	6.1e-05	1.3e-08	4.8e+03
5	6.1e-05	1.9e-09	3.3e+04
6	6.1e-05	2.6e-10	2.3e+05
7	6.1e-05	3.7e-11	1.7e+06
8	6.1e-05	5.1e-12	1.2e+07
9	6.1e-05	6.9e-13	8.9e+07
10	6.1e-05	9.2e-14	6.6e+08

- [19] M.S. Floater. Evaluation and properties of the derivative of a NURBS curve. *Mathematical Methods in CAGD*, T. Lyche and L.L. Schumaker (eds.), Academic Press, Boston, **2**, pp. 261-274, 1992.
- [20] K.P.S. Gahalaut, J.K. Kraus and S.K. Tomar. Multigrid Methods for Isogeometric Discretization. *Comput. Methods Appl. Mech. Engrg.*, **253**, pp. 413–425, 2013.
- [21] K.P.S. Gahalaut and S.K. Tomar. Condition number estimates for matrices arising in the isogeometric discretizations. *RICAM report*, 23–2012.
- [22] C. Garoni, C. Manni, F. Pelosi, S. Serra-Capizzano and H. Speleers. On the spectrum of stiffness matrices arising from isogeometric analysis. *Numerische Mathematik*. DOI 10.1007/s00211-013-0600-2, 2013.
- [23] W. Hackbusch. *Iterative Solution of Large Sparse Systems of Equations*. Springer, 1994.
- [24] K. Hoellig. Multivariate Splines. *SIAM J. Numer. Anal.* **19(5)**, pp. 1013-1031, 1982.
- [25] K. Hoellig. Stability of B-Spline basis via knot insertion. *Computer Added Geometric Design*. **17**, pp. 447-450, 2000.
- [26] K. Hoellig, U. Reif and J. Wipper. Weighted Extended B-Spline Approximation of Dirichlet Problems. *SIAM J. Numer. Anal.* **39(2)**, pp. 442-462, 2002.
- [27] N. Hu, X-Z Guo, and I.N. Katz. Bounds for eigenvalues and condition numbers in the p -version of the finite element method. *Mathematics of computation*. **67/224**, 1423-1450, 1998.
- [28] T.J.R. Hughes, J.A. Cottrell and Y. Bazilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Comput. Methods Appl. Mech. Engrg.* **194**, 4135-4195, 2005.
- [29] C. Johnson. *Numerical solution of partial differential equations by the finite element method*. Cambridge University Press, Cambridge, New York, 1987.
- [30] T. Lyche, K. Scherer. On the sup-norm condition number of the multivariate triangular Bernstein basis. *Multivariate Approximation and Splines*, G. Nuernberger, J. W. Schmidt, and G. Walz (eds.), ISNM.125, Birkhuser Verlag, Basel. 141-151, 1997.
- [31] T. Lyche, K. Scherer. On the p -norm condition number of the multivariate triangular Bernstein basis *Journal of Computational and Applied Mathematics*. **119**, 259273, 2000.
- [32] J.F. Maitre and O. Pourquier. About the conditioning of matrices in the p -version of the finite element method for second order elliptic problems. *Journal of Computational and Applied Mathematics* **63**, 341-348, 1995.
- [33] J.F. Maitre and O. Pourquier. Condition number and diagonal preconditioning: Comparison of the p -version and the spectral element methods. *Numer. Math.* **74**, pp. 69-84, 1996.
- [34] J.M. Melenk. On condition numbers in hp-FEM with GaussLobatto-based shape functions. *Journal of Computational and Applied Mathematics. Volume 139, Issue 1*. 21-48, 2002.
- [35] J.M. Melenk and I. Babuska. The Partition of Unity Finite Element Method: Basic Theory and Applications. *Comput. Methods Appl. Mech. Engrg.* **139**, pp. 289-314, 1996.
- [36] B. Mössner and U. Reif. Stability of tensor product B-Splines on domains. *J. Approx. Theory* **154**, 1-19, 2008.
- [37] E.T. Olsen, J. Douglas, Jr. Bounds on spectral condition numbers of matrices arising in the p -version of the finite element method. *Numerische Mathematik*. **69**, 333-352, 1995.
- [38] J.M. Peña. B-Splines and Optimal Stability. *Mathematics of Computation* **66**, pp. 1555-1560, 1997.
- [39] L. Piegl and W. Tiller. *The NURBS Book (Monographs in Visual Communication)*, Second ed.,

- Springer-Verlag, 1997.
- [40] E. Pilgerstorfer, B. Jüttler. Bounding the influence of domain parameterization and knot spacing on numerical stability in Isogeometric Analysis. *Computer Methods in Applied Mechanics and Engineering*. **268**, pp. 589–613, 2014.
 - [41] P.M. Prenter. *Splines and Variational Methods*. John Wiley & Sons, New York, 1975.
 - [42] D.F. Rogers. *An Introduction to NURBS With Historical Perspective*. Academic Press, 2001.
 - [43] Y. Saad. *Iterative Methods for Sparse Linear Systems*. PWS Publishing Company, Boston, 1996.
 - [44] K. Scherer, A.Y. Shadrin. New upper bound for the B-Spline basis condition number, I. *East J. Approx.* **2**, 331-342, 1996.
 - [45] K. Scherer, A.Y. Shadrin. New upper bound for the B-Spline basis condition number, II. A proof of de Boor's 2^k -conjecture. *Journal of Approximation theory*. **99**, 217-229, 1999.
 - [46] M.H. Schultz. *Spline Analysis*. Prentice-Hall, Englewood Cliffs, 1973.
 - [47] L.L. Schumaker. *Spline Functions: Basic Theory*. Cambridge University Press, 2007.
 - [48] G. Strang and G.J. Fix. *An Analysis of the Finite Element Method*. Prentice Hall, 1973.
 - [49] R.S. Varga. *Matrix Iterative Analysis*. Prentice Hall, New Jersey, 1965.
 - [50] A. Weiser, S.C. Eisenstat, and M.H. Schultz. On solving elliptic equations to moderate accuracy. *SIAM J. Numer. Anal.* **17**, 908-929, 1980.